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An Energy Conservative Ray-Tracing Method With a Time Interpolation of the Force Field

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An Energy Conservative Ray-Tracing Method With A Time Interpolation Of The Force-Field

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Abstract

A new algorithm that constructs a continuous force field interpolated in time is proposed for resolving existing difficulties in numerical methods for ray-tracing. This new method has improved accuracy, but with the same degree of algebraic complexity compared to Kaisers method[1].

INTRODUCTION

A new ray-tracing algorithm that utilizes time-dependent interpolation of the force in a differentiable potential field is proposed. The new method uses known boundary values of force terms to constrain the solution, therefore is of order higher accuracy compared to Kaiser's method [1] and has the same degree of numerical complexity. The new method also provides continuation of force and an exact energy conservation where the ray travels across cell boundaries. When the ray reaches the critical surface, the turning of ray is treated naturally. Furthermore, because of continuity of the force field constructed with the new method, Snell's Law required by the Kaiser's method to be applied at cell boundary when jump of potential field occurs is not required with the new method. Thus, no ray reflecting/splitting will occur with the new method. Numerical examples show the new method generates promising ray-tracing solutions with excellent agreement to the exact solution for a nonlinear potential field.

With the proposed new method, the force that a ray experienced while traveling through a cell is not a constant, but linearly interpolated with the times when the ray intersects the cell boundary, and is consistent with their values on cell boundary, thus is continuous across cell boundaries. The time for a ray to

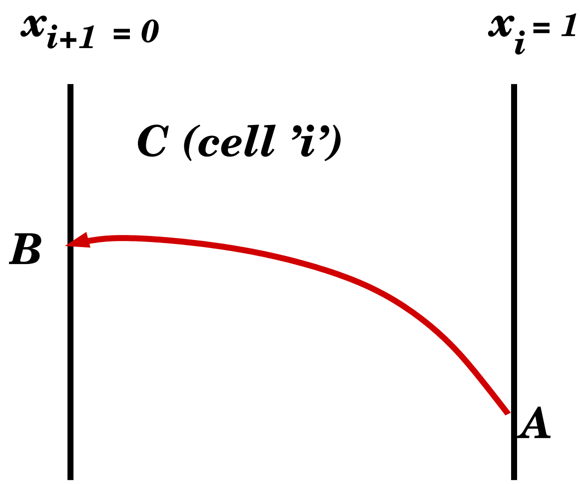

Fig. 1. *Ray-tracing between a pair of cell walls (a single cell).*

travel travel through a given cell is back-solved with and energy conservation at the exiting point and the time integrated position at it.

In this report, the mathematical approach with proposed method is explained at first. After that, numerical examples are given in one-dimensional geometry to demonstrate the improved accuracy of the proposed method over the method in[1]. Compared to the analytical solution for ray-tracing in a given nonlinear force field, with varying the initial direction of the ray, keeping the initial speed to *one*, the new method has shown more accurate solutions with time traveled by the ray and the position of the trajectory of the ray. Better agreement to the exact solution in the case that the ray turns back in the potential field is also shown.

THE PROPOSED METHOD vs. KAISER'S METHOD

The trace of a particle with a unity mass (a ray) traveling in a potential field $V(\vec{r})$ is being considered here. Assuming the ray enters a cell **C** from point **A** with velocity $\vec{u}_A$ and exits at point **B** with the velocity $\vec{u}_B$, we are going to derive the trace of the ray and examine energy conservation where the ray enters the next cell.

The equation of motion of the particle (or ray) is by Newton's second law that

$$\frac{d^2\vec{r}}{dt^2} = \vec{g}(\vec{r}) = -\vec{\nabla}V(\vec{r}). \tag{1}$$

where $\vec{g}$ is gravity (the force), the gradient of the field potential $V(\vec{r})$. A first integral is easy to obtain, which is the energy conservation law that

$$\frac{1}{2}u^2 + V(\vec{r}) = const. \tag{2}$$

With Kaiser's method, a constant force $\vec{g}$ (gradient of the potential) is assumed inside cell **C** and the solution is

$$\vec{u} = \vec{u}_A + \vec{g}t,$$

$$\vec{r} = \vec{r}_A + \vec{u_0}t + \frac{1}{2}\vec{g}t^2.$$

Let T be the time it takes for the ray to reach the exiting point **B**, then at point **B**, one has

$$\vec{u}_B = \vec{u}_A + \vec{g}T,$$

$$\vec{r}_B = \vec{r}_A + \vec{u_A}T + \frac{1}{2}\vec{g}T^2.$$

However, with the proposed method, the force is an interpolation of its values at points **A** and **B** and the force at cell center

$$\vec{g} = (1 - \frac{2t}{T})\vec{g}_A + \frac{2t}{T}(1+\delta)\vec{g}_c, \tag{3}$$

for $0 \leq t \leq T/2$, and

$$\vec{g} = 2(1 - \frac{t}{T})(1+\delta)\vec{g}_c + (\frac{2t}{T} - 1)\vec{g}_B, \tag{4}$$

for $T/2 \leq t \leq 1$.

Where the term $(1+\delta)\vec{g}_c$ is the force applied to the ray at the half-travel time $T/2$, δ is considered as a correction to the force at cell center because at half time, a ray does not necessarily pass the cell center. δ is to be determined with checking energy conservation at cell boundary.

A time integration of the governing equations for the first half-travel time provides

$$\vec{r} = \vec{r}_A + \vec{u}_A t + \vec{g}_A(\frac{t^2}{2} - \frac{t^3}{3T}) + \frac{1+\delta}{3}\vec{g}_c\frac{t^3}{T}, \tag{5}$$

$$\vec{u} = \vec{u}_A + \vec{g}_A t(1 - \frac{t}{T}) + \frac{1+\delta}{T}\vec{g}_c t^2.$$

At half-time $t = T/2$, one has

$$\vec{r}_{T/2} = \vec{r}_A + \frac{\vec{u}_A}{2}T + \frac{1}{12}\vec{g}_A T^2 + \frac{1+\delta}{24}\vec{g}_c T^2,$$

$$\vec{u}_{T/2} = \vec{u}_a + \frac{\vec{g}_A}{4}T + \frac{1+\delta}{4}\vec{g}_c T.$$

Then we are able to integrate for $T/2 \le t \le T$ and obtain

$$\vec{r}_B = \vec{r}_T = \vec{r}_A + \vec{u}_A T + \frac{5\vec{g}_A + \vec{g}_B}{24}T^2 + \frac{1+\delta}{4}\vec{g}_c T^2, \tag{6}$$

$$\vec{u}_B = \vec{u}_T = \frac{\vec{g}_A + \vec{g}_B}{4} + \frac{1+\delta}{2}\vec{g}_c + \vec{u}_a.$$

Now we take a look of energy conservation at r_B, we have the total dynamical energy

$$e_A = \frac{|u_A|^2}{2} + V_A, e_B = \frac{|u_B|^2}{2} + V_B,$$

enforcing energy conservation $e_A - e_B = 0$ generates the equation for (T, δ) that

$$(\vec{g}_A + \vec{g}_B + 2(1+\delta)\vec{g}_c)[8\vec{u}_A + (\vec{g}_A + \vec{g}_B + 2(1+\delta)\vec{g}_c)T] = 32(V_A - V_B).$$

This equation, coupled with other conditions at $\vec{r}_B$ and $\vec{r}_{T/2}$, will determine the solution of the system. For example, if $\vec{r}_B$ is on a bi-linear face defined by $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4$, then the monotonic parametric interpolation $\vec{r}_B = \Sigma_k \phi_k(r,s)\vec{r}_k$, $V_B = \Sigma_k \phi_k(r,s)V_k$ (r, s are the parametric coordinate) will provide additional equations for a closure of the algebraic system with variables (r, s, δ, T).

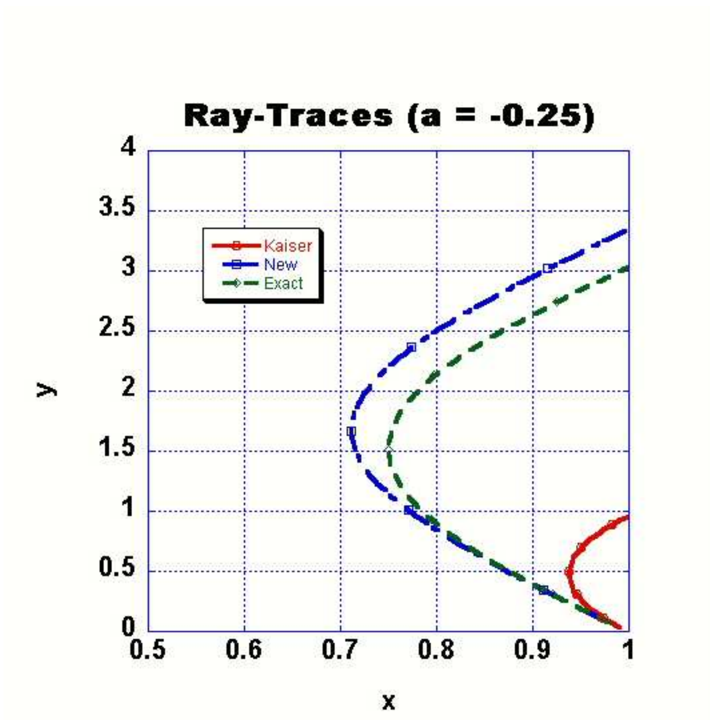

Fig. 2. *Ray-tracing between a pair of cell walls (a single cell) with* $a = 0.25$*,* $\alpha = 2$*.*

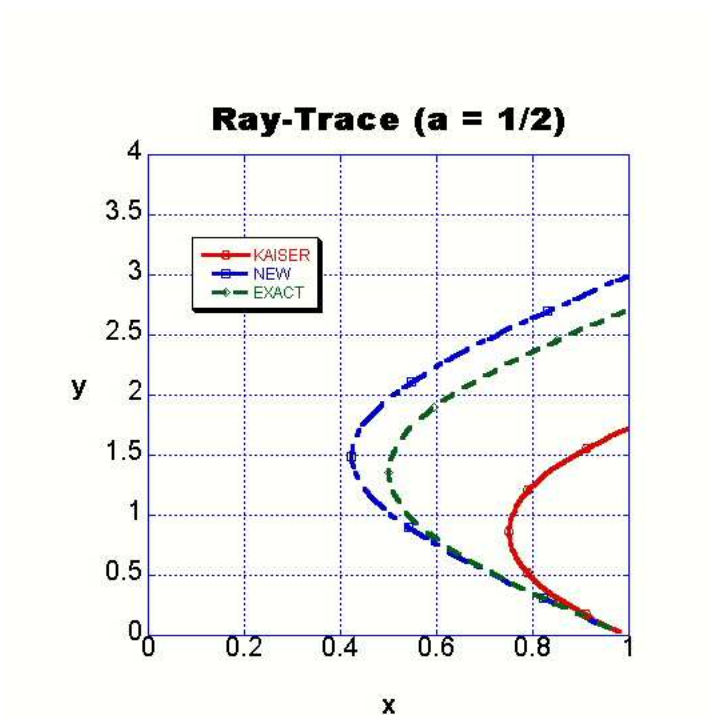

Fig. 3. *Ray-tracing between a pair of cell walls (a single cell) with* $a = 0.50$*,* $\alpha = 2$*.*

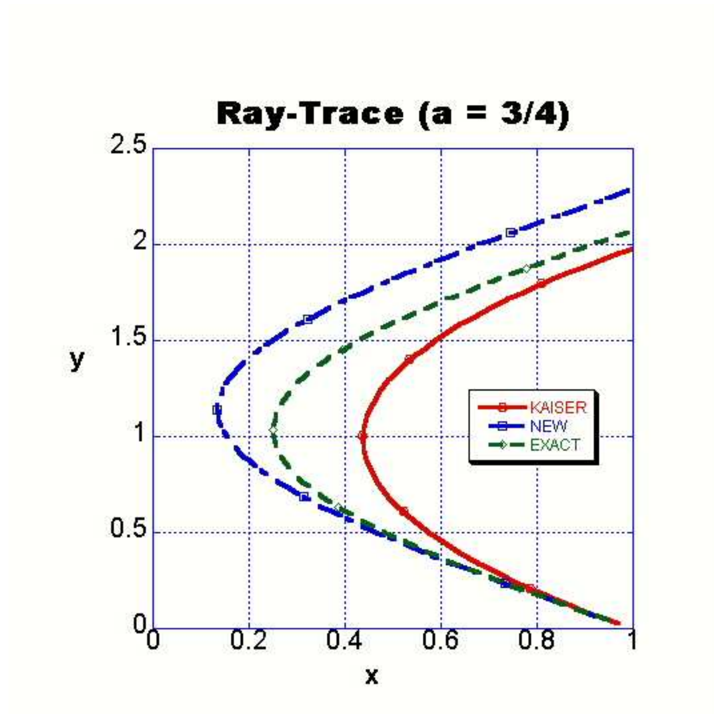

Fig. 4. *Ray-tracing between a pair of cell walls (a single cell) with* $a = 0.75, \alpha = 2$.

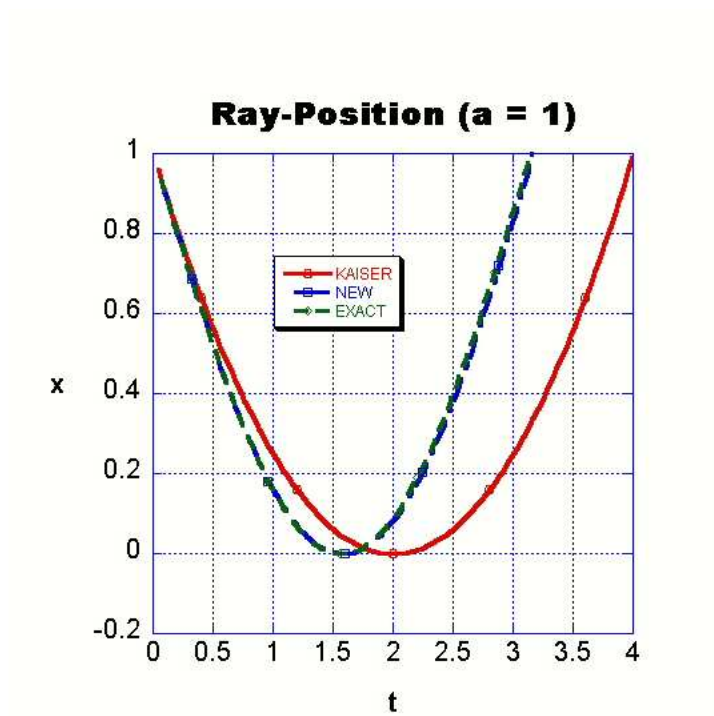

Fig. 5. *Time history of ray-tracing between a pair of cell walls (a single cell) with* $a = 1, \alpha = 2$. *Solution of ray-trace with the new method almost overlaps the exact solution.*

NUMERICAL TESTS

The test problem we have chosen is ray-tracing in a potential field varying in the x-direction

$$V(\vec{r}) = \frac{1}{2}(1-x)^{\alpha}, (\alpha > 1, x \geq 0), \tag{7}$$

when $0 \leq x \geq 1$, $V = 0$ when $x > 1$. The mesh is one-dimensional and set between $[0, 1]$ in the x-direction. The ray enters the mesh at point $A : \vec{r}_0 = (1, 0)$ with velocity $\vec{u}_0 = (-a, b)$ where $a^2 + b^2 = 1, (a, b > 0)$. Therefore, the total dynamical energy of the ray (considered as a particle of unit mass) is 1/2 because at the entering point A, the potential energy is *zero*.

The force (acceleration or gravitation) term can be derived by direct differentiation

$$\vec{f}(\vec{r}) = -\vec{\nabla}V(\vec{r}) = \frac{1}{2}\alpha(1-x)^{\alpha-1}\hat{i}.$$

The above problem can be exactly integrated in some simple cases.

For $\alpha = 1$, one has

$$x = 1 - at + \frac{1}{4}t^2, \qquad y = bt. \tag{8}$$

The time take to travel in the potential field is $T = 4a$, and the exiting point will be at $B : \vec{r}_1 = (1, 4ab)$, with the exiting velocity $\vec{u}_B = (a, b)$.

For $\alpha = 2$, one has

$$x = 1 - a \times sin(t), \qquad y = bt. \tag{9}$$

The time takes to travel through the potential field is $T = \pi$ (if the ray does not make a turn, and arrives the other side), and the exiting point will be at $B : \vec{r}_1 = (1, \pi b)$ with the exiting velocity $\vec{u}_B = (a, b)$.

To start with, let there be N cells in the x-direction, with evenly spaced mesh lines between $(0, 1)$. The width of each cell is $\Delta x = 1/N$. For cell $i, (i = 1, 2, 3, ...N)$, its left face is $x_{left} = (i-1)\Delta x$ and its right face is $x_{right} = i\Delta x$. For the purpose of demonstration, we discuss the most simple cases of $N = 1$ and $N = 2$.

We express the force $\vec{g}$ with a linear interpolation of theoretical values at boundary. The case of constant force with $\alpha = 1$ is trivial, both Kaiser's method and the new method exactly agrees with the theory. We will pay attention to the nontrivial case $\alpha = 2$.

WITH A SINGLE CELL $N = 1$

First of all, we let $N = 1$ and compare Kaiser's solution and new solution with the exact solution. In this case the variation of density gradient is $O(1)$, the numerical methods are supposed to be valid only when the said variation is $o(1)$.

KAISER's METHOD

Kaiser's method assumes a constant force. In this case which is expressed by $-(V(1) - V(0))\hat{\imath}$ divided by the cell width 1. Therefore, $\vec{g} = \hat{\imath}/2$ for $\alpha = 2$, the value we choose. This force vanishes in the case $\alpha = 1$. The ray then arrives at the other side of cell at the time $T = 4a$, and the exiting point $\vec{r}_1 = (1, 4ab)$, with an exit velocity $\vec{u}_1 = (a, b)$.

THE NEW METHOD

We assume the boundary value of the force $\vec{g}$ is specified. The force in the cell center is obtained with a linear interpolation of boundary values. Again, we obtain $\vec{g}_c = \hat{\imath}/2$ for $\alpha = 2$. The equation of motion (5, 6) the become (with $\vec{g}_B = 0$.)

$$\vec{g} = (1 - \frac{2t}{T})\vec{g}_A + \frac{2t}{T}(1+\delta)\vec{g}_c = \frac{t}{T}(1+\delta)\hat{\imath}, \tag{10}$$

for $0 \le t \le T/2$, and

$$\vec{g} = 2(1 - \frac{t}{T})(1+\delta)\vec{g}_c + (\frac{2t}{T} - 1)\vec{g}_B = (1 - \frac{t}{T})(1+\delta)\hat{\imath}, \tag{11}$$

for $T/2 \le t \le 1$. We explicitly assumed that $\vec{g}_B = 0$ by knowing the exiting point has *zero* force. The exiting point on cell boundary is to be determined.

By a direct time integration, we obtain that

$$x_{T/2} = 1 - \frac{a}{2}T + \frac{(1+\delta)}{48}T^2,$$

$$u_{T/2} = -a + \frac{(1+\delta)}{8}T$$

at the mid-point $t = T/2$, and

$$x_{exit} = 1 - aT + \frac{T^2}{8}(1+\delta), \tag{12}$$

$$u_{exit} = -a + \frac{T}{4}(1+\delta).$$

at the exiting point.

Energy conservation at the exiting point requires that $u_{exit} = a$, this is exactly satisfied by observing that $x_{exit} = 1$ gives exactly energy conservation. Next we try to determine δ. Considering that $(1+\delta)\hat{\imath}/2$ be the force applied at $t = T/2$ and is an linear interpolation of forces at boundary, we have a set of equations to solve for T, δ that

$$\frac{(1+\delta)}{2} = \frac{a}{2}T - \frac{(1+\delta)}{48}T^2,$$

for the linearly interpolated force at the turning point, and

$$T(1+\delta) = 8a,$$

for energy conservation at the exiting point.

The algebraic solution is $T = 2\sqrt{3}$, surprisingly, not related to a. The value of T is 3.464102, and is only 10% from the exact solution $T = \pi$. The solution of δ is $4a/\sqrt{3} - 1$. The location of the turning point is $x_{T/2} = 1 - aT/3 = (1 - 1.154701a)$, compared to exact solution $x_{T/2} = (1 - a)$, Kaiser's solution $x_{T/2} = (1 - a^2)$.

The above solution breaks down when $a > \sqrt{3}/2$ for $x_{T/2} < 0$, then we have to consider the possibility for the ray to enter the other cell sharing the wall $x_0 = 0$. With analyzing the special case of $a = 1$, one still has

$$\vec{f} = \frac{t}{T}(1+\delta)\hat{\imath},$$

for $0 \le t \le T/2$. However, for the later half of the path, we have

$$\vec{f} = (1 - \frac{t}{T})(1+\delta)\hat{\imath} + (\frac{2t}{T} - 1)\hat{\imath}.$$

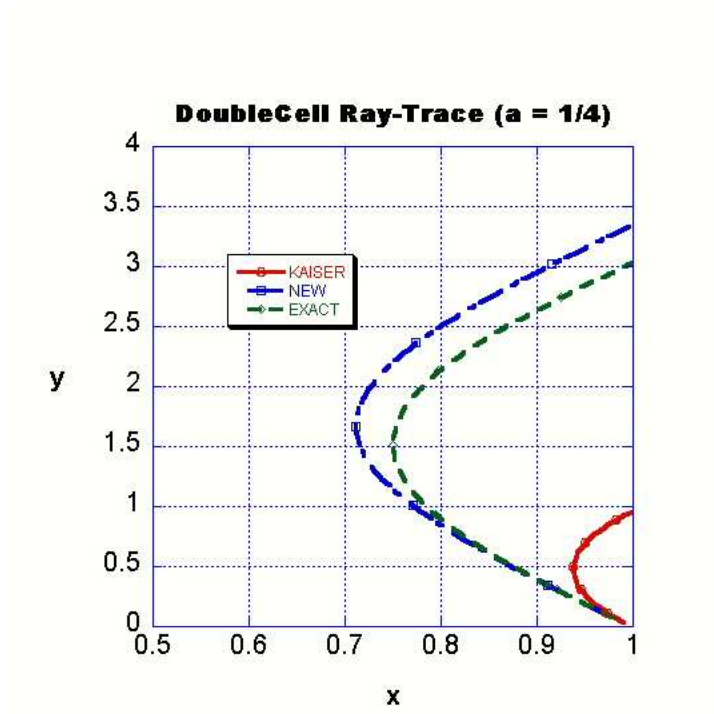

Fig. 6. *Ray-tracing through three cell walls (double cells) with $a = 0.25$, $\alpha = 2$.*

Time integration generates

$$x_T = 1 - T + (\frac{1}{6} + \frac{\delta}{8})T^2,$$

$$u_T = -1 + (\frac{1}{2} + \frac{\delta}{4})T.$$

Because $x_T = 0$ and $u_T = 0$ (from energy conservation), one obtains $T = \sqrt{21} - 3 = 1.58258$. Then because the velocity becomes zero at the cell wall $x_0 = 0$, the ray will reverse its path with symmetry and the total time for the ray to travel through the cell is $t_{exit} = 2T = 3.165160$, which is only 0.75% off the exact solution $t_{exit} = \pi$.

Therefore in the case of $N = 1$, the quality of solution with the new method is much better than Kaiser's solution, not only with good accuracy, but also with exactly conserved energy at the cell boundary and a correct treatment on the critical surface. The ray-traces for the cases with $a = 0.25, 0.5, 0.75$, and 1.0 are plotted in fig.2 through fig.5.

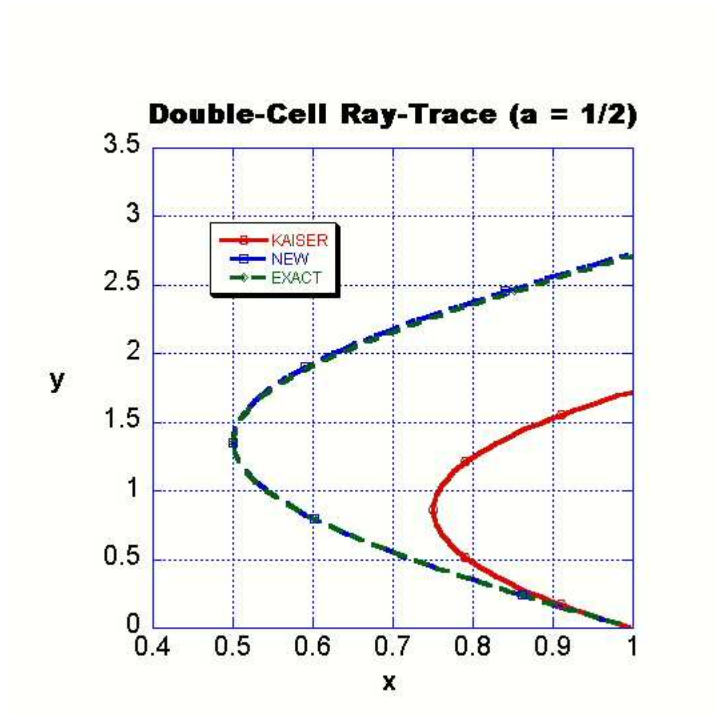

Fig. 7. *Ray-tracing through three cell walls (double cells) with* $\alpha = 0.50, \alpha = 2$.

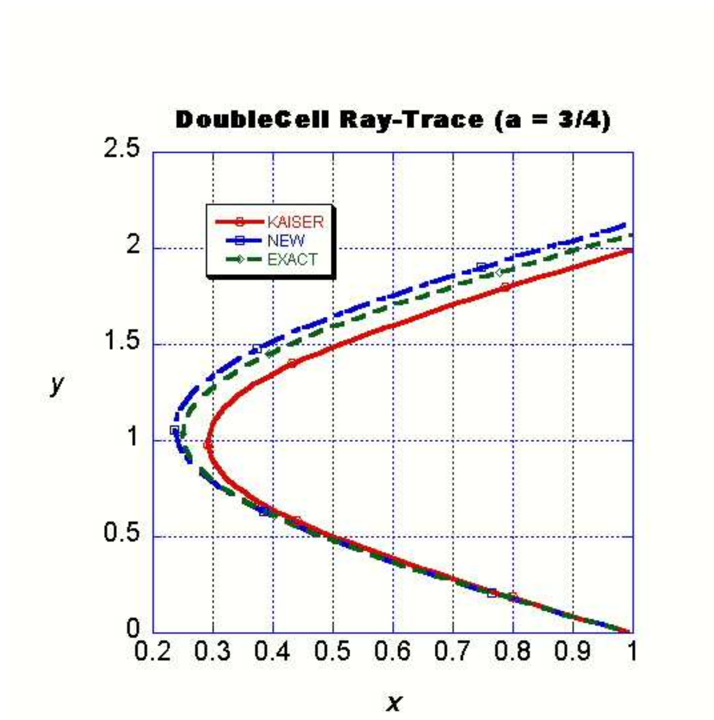

Fig. 8. *Ray-tracing through three cell walls (double cells) with* $\alpha = 0.75, \alpha = 2$.

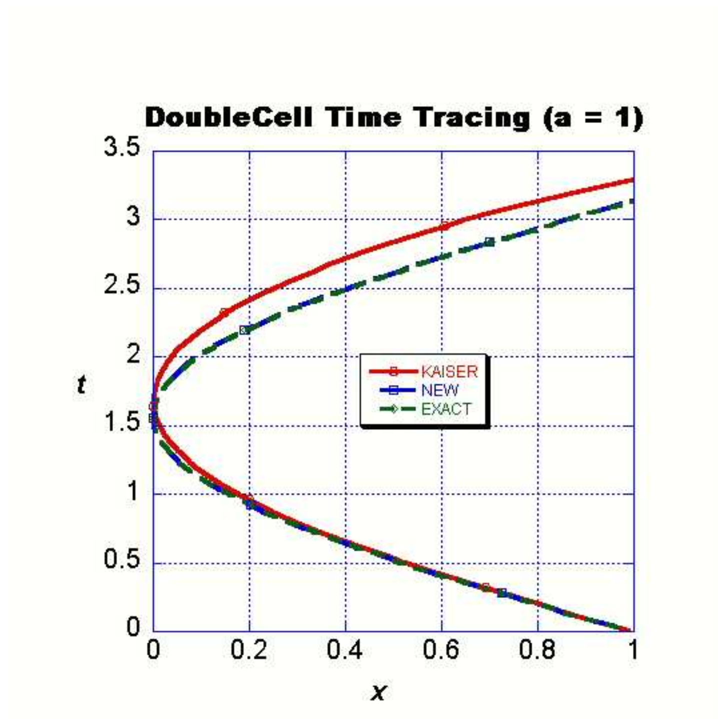

Fig. 9. *Time history for ray-tracing through three cell walls (double cells) with* $a = 1.0, \alpha = 2$. *Solution of ray-trace with the new method almost overlaps the exact solution.*

WITH A PAIR OF CELLS

Then we consider the case $N = 2$ (keeping $\alpha = 2$). For a fair comparison, we again vary a with $four$ values $(1/4, 1/2, 3/4, 1)$ and compare Kaiser's and the new solution in every cases with the exact solution. The positions of cell walls are set to $(x_0, x_1, x_2) = (0, 1/2, 1)$. The corresponding potential at these positions are $(V_0, V_1, V_2) = (1/2, 1/8, 0)$ with $\alpha = 2$. Cell $i (i = 1, 2)$ is defined by (x_{i-1}, x_i).

KAISER's SOLUTION

The ray enters cell 2 first, the force in this cell is obtained by $-(V_2 - V_1)/(x_2 - x_1) = 1/4$. One can directly write down the integral

$$x = 1 - at + \frac{1}{8}t^2, \qquad y = bt. \tag{13}$$

We first examine if there is a turning point in this cell. Since the x- velocity is zero at the possible turning point, one finds that $t = 4a$ at the turning point, therefore $x_t = 1 - 2a^2$, therefore, for $a = 1/4, a = 1/2$, the turning point may be inside cell 2, in the remaining cases, the ray will enter cell 1 after first traveled through cell 2.

Case 1: $a = 1/4$

For $a = 1/4$, the ray will never arrive the cell-wall $x_1 = 1/2$ before exits from $x_2 = 1$, the solution is easily obtained that $t_{turn} = 1$, $x_{turn} = 7/8$, and $y_{turn} = b = \sqrt{15}/4$, then $t_{exit} = 2$, $x_{exit} = 1$, and $y_{exit} = 2b = 2\sqrt{15}/4$.

Case 2: $a = 1/2$

The case $a = 1/2$ is a little bit more interesting because turning point is on the cell boundary $x_1 = 1/2$. The motion of the ray before the turn is governed by

$$x = 1 - \frac{t}{2} + \frac{1}{8}t^2, \qquad y = bt. \tag{14}$$

At the turning point $t_{turn} = 2$, $x_{turn} = 1/2$, and $y_{turn} = bt_{turn} = \sqrt{3}$. Because $dx/dy = 0$ at the turning point, the ray is traveling along the cell boundary, the next cell the ray enters has to be determined by the forces from both cells.

In this case because the force is always positive in $\hat{i}$ direction, the problem is simple and the ray will return to cell 2 with entering velocity $(0, b)$, the ray-trace after turning can be easily integrated to obtain

$$x = \frac{1}{2} + \frac{1}{8}(t-2)^2, \qquad y = bt. \tag{15}$$

When the ray exits the mesh at $x_2 = 1$, we find that $t_{exit} = 4$, thus $y_{exit} = 4b = 2\sqrt{3}$.

Case 3: $a = 3/4$

Our next case is with $a = 3/4$. Kaiser's solution is then

$$x = 1 - \frac{3t}{4} + \frac{1}{8}t^2, \qquad y = bt. \tag{16}$$

At the exiting point $x_1 = 1/2$, one finds $t_{exit} = 3 - \sqrt{5}$, then $y_{exit} = (3-\sqrt{5})b = \frac{\sqrt{7}}{4}(3-\sqrt{5})$, and the exiting velocity is $\vec{u}_{exit} = (-\sqrt{5}/4, \sqrt{7}/4)$. The ray will enter cell 1.

In cell 1, the force is determined by $-(V_1 - V_0)/(x_1 - x_0) = 3\hat{\imath}/4$. Because the mesh is only one-dimensional we are able to smoothly interpolate the values of the potential function with a linear function inside a cell, and there is no jump of potential across the boundary between the two cells.

The solution of ray-trace in cell 1 is then

$$x = \frac{1}{2} - \frac{\sqrt{5}}{4}(t - 3 + \sqrt{5}) + \frac{3}{8}(t - 3 + \sqrt{5})^2, \qquad y = bt. \tag{17}$$

At the turning point , the x-velocity becomes *zero* and we obtain $t_{turn} = 3 - 2\sqrt{5}/3 = 1.509288$, thus $x_{turn} = 7/24$. At the exiting point (on $x_{exit} = x_1 = 1/2$), one has $t_{exit} = 3 - \sqrt{5}/3$, and $\vec{u}_{exit} = (\sqrt{5}/4, \sqrt{7}/4)$.

Then the ray enters cell 2 again. The force in cell 2 is still $\hat{\imath}/2$, then the trace has the solution

$$x = \frac{1}{2} + \frac{\sqrt{5}}{4}(t - 3 + \frac{\sqrt{5}}{3}) + \frac{1}{8}(t - 3 + \frac{\sqrt{5}}{3})^2, \qquad y = bt. \tag{18}$$

When the ray exits the mesh at $x_{exit} = x_2 = 1$, one finds that $t_{exit} = 3 - 2\sqrt{5}/3 + \sqrt{17}/3 = 2.88366$, then $y_{exit} = bt = 1.907362$. The exiting velocity is then $\vec{u}_{exit} = (3/4, \sqrt{7}/4)$.

Case 4: $a = 1$

The final case we decide to examine is $a = 1, b = 0$, the motion of the ray is only one-dimensional. In cell 2, the force is $\hat{\imath}/4$ therefore

$$x = 1 - t + \frac{1}{8}t^2, \qquad y = 0. \tag{19}$$

When ray exits cell 2 at $x_{exit} = x_1 = 1/2$, the travel-time across the cell is $t_{exit} = 4 - 2\sqrt{3}$. The associated velocity is then $\vec{u}_{exit} = (-\sqrt{3}/2, 0)$. Then the ray enters cell 1 to continue travel under force $3\hat{\imath}/4$, the position of the ray is then

$$x = \frac{1}{2} - \frac{\sqrt{3}}{2}(t - t_{exit}) + \frac{3}{8}(t - t_{exit})^2, \qquad y = 0. \tag{20}$$

At the turning point $dx/dt = 0$ and one obtains $t_{turn} = 4(1 - 1/\sqrt{3}) = 1.690599$ and $x_{turn} = 0$. Then the ray starts coming back according to the law of motion

$$x = \frac{3}{8}(t - t_{turn})^2 = \frac{3}{8}(t - 4 + \frac{4}{\sqrt{3}})^2. \tag{21}$$

and arrive at the shared cell boundary $x_1 = 1/2$ at $texit = 4 - 2/\sqrt{3}$, with velocity $\vec{u}_{exit} = (\sqrt{3}/2, 0)$.

Then the ray continues traveling in cell 2 with

$$x = \frac{1}{2} + \frac{\sqrt{3}}{2}(t - 4 + \frac{2}{\sqrt{3}}) + \frac{1}{2}(t - 4 + \frac{2}{\sqrt{3}})^2. \tag{22}$$

When the ray arrives at $x_{final} = x_2 = 1$, one has $t_{final} = 8(1 - 1/\sqrt{3}) = 3.381198$, and the final speed is $u_{final} = 1$, and the travel of ray in the potential field is completed.

The above derivations are for the Kaiser's method for a pair of cells.

THE NEW SOLUTION

To utilize the new method, the values of V and $\vec{\nabla}V$ on cell boundary have to be specified. A practical way to do this is, let V be defined on cell boundary, then use a moving-least-squared operator for a fit of spatially neighboring values of V. In this test case, the base function vector for the least-squared fit can be set to $(1, x, x^2/2)$ to obtain exact function values on the cell boundaries, or set to $(1, x)$ for an accuracy of the second order. To be fair in comparison, we use the rougher choice $(1, x)$. On the mesh line $x_1 = 1/2$, the value of force is consistent with a central difference scheme if the weighting function in the least-squared fit is constant, and we obtain the force at x_1 with $f_1 = -\hat{i}(V_2 - V_0)/(x_2 - x_0) = \hat{i}/2$. On boundary of the mesh, exact value of force is given that $f_0 = \hat{i}$ at $x_0 = 0$, and $f_2 = 0$ at $x_2 = 1$.

Case 1: with a turning point $(a = 1/4)$

We again examine first if there is a turning point in this cell. Because of the symmetry with the problem, we will only examine first half path for possible cases. We recall the new time-interpolated force that

$$\vec{g} = (1 - \frac{2t}{T})\vec{g}_A + \frac{2t}{T}(1 + \delta)\vec{g}_c, \tag{23}$$

for $0 \le t \le T/2$, and

$$\vec{g} = 2(1 - \frac{t}{T})(1 + \delta)\vec{g}_c + (\frac{2t}{T} - 1)\vec{g}_B, \tag{24}$$

for $T/2 \le t \le 1$.

In cell 2, it is natural to take $\vec{g}_c = \hat{i}/4$ for it is the average of boundary values. If we assume the ray turns back without entering cell 1, since force at $x_2 = 1$ is zero, we have

$$\vec{g} = \frac{t}{T}(1+\delta)(\frac{\hat{i}}{2}), \tag{25}$$

for $0 \leq t \leq T/2$, and

$$\vec{g} = (1-\frac{t}{T})(1+\delta)(\frac{\hat{i}}{2}), \tag{26}$$

for $T/2 \leq t \leq T$.

$$x = 1 - at + (1+\delta)\frac{t^3}{12T}, \tag{27}$$

for $0 \leq x \leq T/2$, and

$$x = 1 - at + (1+\delta)(\frac{t^2}{4} - \frac{t^3}{12T} - \frac{tT}{8} + \frac{T^2}{48}), \tag{28}$$

$$u = -a + (1+\delta)(\frac{t}{2} - \frac{t^2}{4T} - \frac{T}{8}).$$

When ray exits at $x_2 = 1$, one has

$$x_T = 1 - \frac{aT}{2} + (1+\delta)\frac{T^2}{16} = 1,$$

$$u_T = -a + (1+\delta)\frac{T}{8}.$$

At exiting point $x_2 = 1$, energy conservation requires $u_T = a$, this is consistent with $x_T = 1$ with $T = 16a/(1+\delta)$.

Then, one finds that $t_{turn} = 8a/(1+\delta)$ and the turning point at $x_{turn} = 1 - 16a^2/3(1+\delta)$. Therefore, at the turning point, the linearly interpolated force will be $\vec{f}_{T/2} = 16a^2/3(1+\delta)$. Then we have $(1+\delta)\hat{i}/4 = 16a^2/3(1+\delta)$ and the solution is $1+\delta = 8a/\sqrt{3}$ and this gives the travel time $T = 2\sqrt{3} = 3.464102$ is not a function of the entering velocity, as long as the turning point is inside cell 2. Therefore, for $a = 1/4$, we have $T = 2\sqrt{3} = 3.464102$ with $x_{turn} = 1 - 2a/\sqrt{3} = 0.711325$ inside cell 2, but for $a \geq \sqrt{3}/4$, the corresponding solution will enter the other cell, after first traveled through cell 2.

Case 2: turning point touches cell wall ($a = 1/2$)

The above analysis does not necessarily mean that when $a = 1/2$ (which is $\geq \sqrt{3}/4$), the ray must enter cell 1, because at this moment the force term is different by means that the force at $x_1 = 1/2$ will contribute to the solution. In the case $a = 1/2$ the force term in (eq. 3) and (eq. 4) will become (with $\vec{g}_a = 0, \vec{g}_B = \hat{\imath}/2$, and $\vec{g}_c = \hat{\imath}/4$)

$$\vec{g} = \frac{t}{T}(1+\delta)\frac{\hat{\imath}}{2}, \tag{29}$$

for $0 \leq t \leq T/2$, and

$$\vec{g} = (1 - \frac{t}{T})(1+\delta)\frac{\hat{\imath}}{2} + (\frac{2t}{T} - 1)\frac{\hat{\imath}}{2}. \tag{30}$$

Time integration for the first half-time is unchanged because of the same force term is applied. Then when the ray reaches the cell wall $x_1 = 1/2$ at $t = T$, one has

$$x_{exit} = 1 - \frac{T}{2} + \frac{T^2}{12} + \frac{\delta}{16}T^2 = x_1 = \frac{1}{2},$$

$$u_{exit} = -\frac{1}{2} + \frac{T}{4} + \frac{\delta}{8}T.$$

Energy conservation at $x_1 = 1/2$ requires that $u_{exit} = 0$, therefore we are able to solve the above system with $T = \sqrt{21} - 3 = 1.582576$, and $\delta = \sqrt{21}/3 - 1 = 0.527525$.

Since $u_T = 0$ and the force is in the $+\hat{\imath}$ direction, the ray actually will not enter cell 1. Instead, the ray will return to cell 2 and finish its travel in the mesh at $t_{total} = 2T = 3.165152$ by symmetry. This solution is vary close to the exact solution that $t_{exact} = \pi = 3.141593$ by an relative error of only 0.75%.

Case 3: turning point in the next cell ($a = 3/4$)

Next we deal with the case that $a = 3/4$, still with the consideration that the energy correction δ might not be zero, one has the force term in cell 2 that

$$\vec{f} = \frac{t}{T}(1+\delta)\frac{\hat{\imath}}{2},$$

for the first half-way, and

$$\vec{f} = \frac{t}{T}\frac{\hat{i}}{2} + \delta(1 - \frac{t}{T})\frac{\hat{i}}{2},$$

for the next half-way.

With this different entering velocity, by time integration one finds that

$$x = 1 - \frac{3}{4}t + \frac{1+\delta}{12T}t^3,$$

and at the end of first half way $t = T/2$, one finds

$$x_{T/2} = 1 - \frac{3}{8}T + (1+\delta)\frac{T^2}{96},$$

$$u_{T/2} = -\frac{3}{4} + \frac{T}{16}(1+\delta).$$

At the exiting point $t_{exit} = T$, $x_{exit} = x_1 = 1/2$ one has

$$x_{exit} = 1 - \frac{3}{4}T + \frac{T^2}{12} + \frac{\delta}{16}T^2,$$

$$u_{exit} = -\frac{3}{4} + \frac{T}{4} + \frac{\delta}{8}T.$$

With energy conservation at $x_1 = 1/2$, let $\lambda = T/4 + \delta T/8$, then one finds $\lambda^2 - 3\lambda/2 + 1/4 = 0$, then $\lambda = (3 - \sqrt{5})/4$, and which makes $u_{exit} = -\sqrt{5}/4$. Because

$$x_{exit} - u_{exit}\frac{T}{2} = 1 - \frac{3}{8}T - \frac{T^2}{24},$$

this quadratic equation for T gives that $T = 0.730006$, $\delta = 0.0929472$. Note that the exact solution $x = 1 - a \times sin(t)$ gives $T = sin^{-1}(2/3) = 0.729728$, we see the relative error with the new method is only about 0.03%.

The ray then enters cell 1 with $\vec{u} = (-\sqrt{5}/4, \sqrt{7}/4)$. We assume that the ray will make a turn in this cell then the force term with the new method applied to this portion of ray-path is (with $\vec{f_1} = \hat{i}/2$ the force on the cell wall $x_1 = 1/2$ and $f_c = (\hat{i} + \hat{i}/2)/2 = 3\hat{i}/4$ the average force in cell 1.

$$\vec{f} = (1 - 2\frac{t}{T})\frac{\hat{i}}{2} + 2\frac{t}{T}(1+\delta)(\frac{3}{4}\hat{i}),$$

for the first half-way, and

$$\vec{f} = 2(1 - \frac{t}{T})(1+\delta)(\frac{3}{4}\hat{i}) + (2\frac{t}{T} - 1)\frac{\hat{i}}{2},$$

for the next half-way.

The solution for $0 \le t \le T/2$ is

$$x = \frac{1}{2} - \frac{\sqrt{5}}{4}t + \frac{t^2}{4} + \frac{t^3}{T}(\frac{1}{12} + \frac{\delta}{4}),$$

$$u = -\frac{\sqrt{5}}{4} + \frac{t}{2} + \frac{t^2}{T}(\frac{1}{4} + \frac{3}{4}\delta).$$

At the turning point $u = 0$ (or $t = T/2$ by symmetry) and one has

$$x_{turn} = \frac{1}{2} - \frac{\sqrt{5}}{8}T + \frac{7}{96}T^2 + \frac{\delta}{32}T^2,$$

$$u_{turn} = -\frac{\sqrt{5}}{4} + \frac{T}{16}(5 + 3\delta) = 0.$$

then the solution after turning when the ray return to cell wall $x_1 = 1/2$ will be

$$x_T = \frac{1}{2} - \frac{\sqrt{5}}{8}T + \frac{5}{32}T^2 + \frac{3}{32}\delta T^2,$$

$$u_T = \frac{5}{16}T + \frac{3}{16}\delta T.$$

The energy conservation requires that $u_T = \sqrt{5}/4$. Then one observes that it exactly gives $x_T = 1/2$, and $u = 0$ at the turning point. Therefore the energy conservation is not affected by the choice of δ, and we may choose $\delta = 0$, and have $T = 4/\sqrt{5} = 1.788854$. Be aware that T here is the time takes for the ray to travel through cell 1 only. Then the position of the turning point is at $x_{turn} = 7/30$ and the interpolated force there will be $[(7/30)\vec{f_1} + (8/30)\vec{f_0}]/(x_1 - x_0) = (23/30)\hat{i} = 0.7666667\hat{i}$, therefore at $t = T/2$, we have a more accurate estimate of the force than $3/4\hat{i} = 0.75\hat{i}$. Insert this value for $\vec{f_c}$ in the force term, one has

$$\vec{f} = (1 - 2\frac{t}{T})\frac{\hat{i}}{2} + 2\frac{t}{T}(\frac{23}{30}i),$$

for the first half-way, and

$$\vec{f} = 2(1 - \frac{t}{T})(\frac{23}{30}\hat{i}) + (2\frac{t}{T} - 1)\frac{\hat{i}}{2},$$

for the next half-way.

Integrate the force terms through the turning point, one finds

$$u_{turn} = -\frac{\sqrt{5}}{4} + \frac{19}{60}T = 0.$$

Therefore an improved $T = 15\sqrt{5}/19 = 1.765317$ is obtained.

Then by symmetry, the motion after the ray re-enters cell 2 will exactly reverse the solution before the ray enters cell 1 (in x- direction). We know the time it takes will be 0.730006 and the total time of the travel will be $0.730006 + 1.765317 + 0.730006 = 3.225329$, which is only about 2.7% from the exact solution $t_{total} = \pi$.

Case 4: the critical surface ($a = 1$)

The last case we decide to test is with $a = 1$. This case is simple because the motion is only in the x-direction.

We write down the force term in cell 2 similarly to the case of $a = 3/4$ that

$$\vec{f} = \frac{t}{T}(1 + \delta)\frac{\hat{i}}{2},$$

for the first half-way, and

$$\vec{f} = \frac{t}{T}\frac{\hat{i}}{2} + \delta(1 - \frac{t}{T})\frac{\hat{i}}{2},$$

for the next half-way.

With the entering velocity be $(-1, 0)$, by time integration one finds that

$$x = 1 - t + \frac{1 + \delta}{12T}t^3,$$

and at the end of first half way $t = T/2$, one finds

$$x_{T/2} = 1 - \frac{1}{2}T + (1+\delta)\frac{T^2}{96},$$

$$u_{T/2} = -1 + \frac{T}{16}(1+\delta).$$

At the exiting point $t_{exit} = T$, $x_{exit} = x_1 = 1/2$ one has

$$x_{exit} = 1 - T + \frac{T^2}{12} + \frac{\delta}{16}T^2,$$

$$u_{exit} = -1 + \frac{T}{4} + \frac{\delta}{8}T.$$

Energy conservation at $x_1 = 1/2$ requires that $u_{exit} = -\sqrt{3}/2$ and the solution is given by $T = 0.523653$ and $\delta = 0.0467936$. Note that the exact solution is $T = sin^{-1}(1/2) = \pi/6 = 0.523599$ and the relative error is only 0.01%.

Then the ray enters cell 1 with velocity $u = -\sqrt{3}/2$, the force term will be

$$\vec{f} = (1 - 2\frac{t}{T})\frac{\hat{i}}{2} + 2\frac{t}{T}(1+\delta)(\frac{3}{4}\hat{i}),$$

for the first half-way, and

$$\vec{f} = 2(1 - \frac{t}{T})(1+\delta)(\frac{3}{4}\hat{i}) + (2\frac{t}{T} - 1)\frac{\hat{i}}{2},$$

for the next half-way.

Because the ray will turn back to $x_1 = 1/2$, the choice of δ shall not affect energy conservation. We integrate the force term to the turning point ($t = T/2$) and obtain

$$x_{turn} = \frac{1}{2} - \frac{\sqrt{3}}{4}T + \frac{7}{96}T^2,$$

$$u_{turn} = -\frac{\sqrt{3}}{2} + \frac{5}{16}T = 0.$$

Then obtain $T = \frac{8}{5}\sqrt{3} = 2.771281$, $x_{turn} = -7/50 = -0.14$. This means that the assumption of a turning point under the given force term is invalid and the ray is going to arrive $x_0 = 0$ under such a force.

Then the force term has to include contribution from $f_0 = \hat{i}$ with

$$\vec{f} = (1 - 2\frac{t}{T})\frac{\hat{i}}{2} + 2\frac{t}{T}(1+\delta)(\frac{3}{4}\hat{i}),$$

for the first half-way, and

$$\vec{f} = 2(1 - \frac{t}{T})(1+\delta)(\frac{3}{4}\hat{i}) + (2\frac{t}{T} - 1)\hat{i},$$

for the next half-way. T here is the time takes for the ray to cross cell 1. Time integration to $t = T$ gives us

$$x_T = \frac{1}{2} - \frac{\sqrt{3}}{2}T + \frac{T^2}{3} + \frac{3}{16}\delta T^2 = x_0 = 0,$$

$$u_T = -\frac{\sqrt{3}}{2} + \frac{3}{4}T + \frac{3}{8}\delta T.$$

At $x_T = 0$ because the potential $V_0 = 1/2$, one has to have $u_T = 0$ to satisfy energy conservation, thus

$$T = (\sqrt{13} - 3)\sqrt{3} = 1.048846,$$

$$\delta = \frac{22 - 6\sqrt{13}}{3(\sqrt{13} - 3)} = 0.20185.$$

Because at $x = x_0 = 0$, $u = 0$ there fore the location is the turning point and the later half ray-path exactly reverse the first half. Therefore the total time for the ray to travel through cell 1 is $t_{exit} = 2T = 2.097691$. Then the total time for the ray to travel through the potential field will be $t_{total} = 0.523653 + 2.097691 + 0.523653 = 3.144997$. Compared to the exact solution $t_{total} = \pi$, the relative error is only 0.1%.

The above derivations are for the proposed new method for a pair of cells.

COMPARISON BETWEEN THE NEW METHOD AND THE KAISER'S METHOD

The comparisons between the methods are explicitly listed in numbers in this section. For energy disposition, the time it takes for a ray to travel through a cell is important. Table 1 and 2 show the results of the travel time calculated for the cases with a single cell; and with a pair of cells. The location of the turning point indicates how close the numerical ray-traces are to the exact solution in geometry. For the case of a single cell, formulas can be given; for the case of double cells, the location of turning point is shown in table 3. Plots of ray-traces are also provided.

In the case of a single cell, the relative time error to the exact solution calculated with different methods varying entering velocities are shown in the following table

a	$(T, E)_{Kaiser}$	$(T, E)_{new}$	T_{exact}
1/4	(1, 314%)	(3.414602, 10.2 %)	3.141593
1/2	(2, 157%)	(3.414602, 10.2 %)	3.141593
3/4	(3, 4.7%)	(3.414602, 10.2 %)	3.141593
1	(4, 27.3%)	(3.165160, 0.75%)	3.141593

Table 1. Travel time through a single cell.

Formula with each method for the x-coordinate of the turning point with a single cell is the follows: $(1 - a^2)$ (Kaiser); $(1 - 1.154701a)$ (new method, for $a < \sqrt{3}/2$); and $(1 - a)$ (exact).

In the case of a pair of cells, the ray travel-time and its relative error to the exact solution, varying entering velocities is shown in the following table

a	$(T, E)_{Kaiser}$	$(T, E)_{new}$	T_{exact}
1/4	(2, 157%)	(3.464102, 10.2 %)	3.141593
1/2	(4, 27.3%)	(3.165152, 0.75 %)	3.141593
3/4	(2.88366, 8.9%)	(3.225329, 2.7 %)	3.141593
1	(3.381198, 7.6%)	(3.144997, 0.1%)	3.141593

Table 2. Travel time through double cells.

The position of the turning point in the case of a single cell is listed in the following table

The ray traces calculated from Kaiser's method, the new method, and the exact solutions with two cells are plotted in fig.5 through fig.9.

a	$(x_{turn}, error)_{Kaiser}$	$(x_{turn}, error)_{new}$	x^{turn}_{exact}
1/4	(0.875, 16.7%)	(0.711325, 5.1 %)	3/4
1/2	(0.5, 0%)	(0.5, 0%)	1/2
3/4	(0.291667, 16.7%)	(0.233333, 6.7 %)	1/4
1	(0, NA)	(0, NA)	0

Table 3. Location of turning point with double cells.

CONCLUSION

The proposed new ray-tracing method, takes the energy conservation on cell boundary in account, provides a much more accurate solution compared to the bench-mark Kaiser's method. A continuous potential field is constructed with smooth interpolation of boundary potential values. This continuity allows us to avoid using the Snell's law on cell boundaries thus eliminates the reflecting/splitting when a ray passed cell boundaries. Although we demonstrated the advantage of the new method with only one-dimensional cells, there is no obstacle in sight to generate the method to higher dimension problems. Because the field in interpolated in time with boundary values, the continuity of force across multi-dimensional cell boundaries is again insured.

We conclude that the proposed new method has advantages over the Kaiser's method in providing quality solution of a ray-tracing problem in general.

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